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Gauge theory amplitudes in twistor space and holomorphic anomaly

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Abstract: We show that, in analyzing differential equations obeyed by one-loop gauge theory amplitudes, one must take into account a certain holomorphic anomaly. When this is done, the results are consistent with the simplest twistor-space picture of the available one-loop amplitudes.

Keywords: Gauge Symmetry, Extended Supersymmetry, NLO Computations.

Contents

1. Introduction

Perturbative scattering amplitudes of gluons in Yang-Mills theory exhibit many remarkable properties, some of which [1] apparently reflect the fact that they can be computed using a string theory in which twistor space is the target space.

At tree level, the amplitudes can be constructed either from connected diagrams in twistor space, as described most fully in [2], or from disconnected diagrams [3]. In the latter approach, one constructs the tree amplitudes from Feynman diagrams in which the vertices are tree-level MHV (maximal helicity violating) amplitudes, continued off-shell, and the propagators are simple Feynman propagators $1/p^2$. For a discussion of the relation between these two approaches, see [4].

At one-loop, the twistor space structure of various gauge theory amplitudes was studied in [5] by studying the differential equations that they obey. The simple twistor space picture of [1] suggests that the one-loop MHV amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory are a sum of terms corresponding to the twistor space pictures of figures $1a$ and $1b$ — all gluons supported on a pair of lines (figure $1a$) or on a degree two curve of genus zero (figure $1b$). However, the differential equations appear to indicate the presence of a further contribution (figure $1c$) in which all gluons but one are on a pair of lines, while the remaining gluon is coplanar with the two lines.[1] This is perplexing, because there has been no proposal for a twistor-string mechanism to generate a contribution with this structure.

We have reconsidered these issues because of a new computation of the one-loop $\mathcal{N} = 4$ MHV amplitudes [6]. In that computation, the full amplitude is obtained from a one-loop amplitude with two MHV vertices and $1/p^2$ propagators (figure 2); this is a direct one-loop generalization of the tree diagrams considered in [3]. Since each MHV vertex is supported on a line in twistor space, this computation seems to make it manifest that these one-loop MHV amplitudes are supported on a pair of lines, the configuration of figure $1a$. The configuration of figure $1b$ is also possible (when one propagator collapses) given that the degree two curve in figure $1b$ is really a pair of intersecting lines, as mentioned in the last footnote. But a configuration of the type of figure $1c$ seems to be missing.

[1] The differential equations seem to indicate that the degree two curve of figure $1b$ degenerates to a pair of intersecting lines, as is drawn in the figure. This behavior is not entirely understood, and needs to be interpreted with care, as explained in [5].

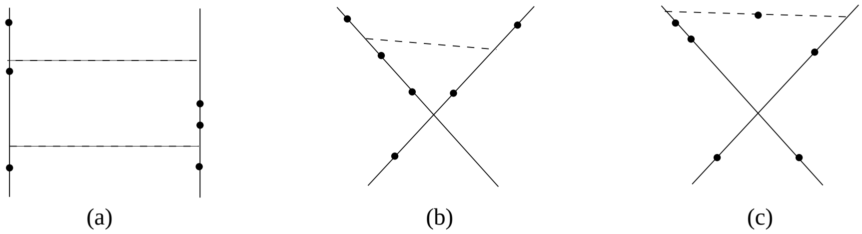

Figure 1: Shown here are twistor configurations that were found in [5] to contribute to one-loop supersymmetric MHV amplitudes. In (a), all gluons are inserted on a pair of possibly disjoint lines connected by two twistor space propagators. In (b), all gluons are inserted on a degree two curve of genus zero with a twistor space propagator; the curve is here drawn simply as a pair of intersecting lines. (c) is just like (b) except that one gluon is inserted not on the pair of intersecting lines but somewhere else in the plane containing the two lines. In the figures, dashed lines indicate twistor space propagators.

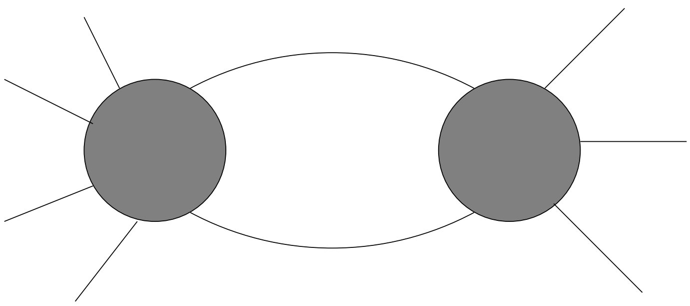

Figure 2: Diagram contributing to a one-loop $\mathcal{N} = 4$ MHV amplitude. Each disc represents a "line," that is a $\mathbb{CP}^1$ in twistor space, which generates a tree-level MHV amplitude with gluons attached. The loop amplitude is computed by connecting two such MHV vertices via exchange of two gluons.

This problem can be posed more sharply if one considers the imaginary part of the scattering amplitude. Even without invoking the full cut-constructibility of these one-loop amplitudes [7], one knows just from unitarity that for real momenta in Lorentz signature, the imaginary part of the scattering amplitude comes from a sum over on-shell intermediate states. Thus, the imaginary part of the amplitude can be obtained from the "cut" diagram of figure 3, where the "cut" propagators are on-shell and the scattering amplitudes on the left and right are on-shell tree level MHV amplitudes. This seems to show that at least the discontinuities (or on-shell imaginary parts) of the scattering amplitudes must be supported on a pair of lines. However, when we investigate the differential equations obeyed by the imaginary part of the scattering amplitudes, we find (as one would guess from the fact that these amplitudes can be constructed from their four-dimensional cuts) that the imaginary parts obey the same differential equations as the full amplitudes; they do not obey additional equations which would assert the absence of a contribution of the type of figure 1c.

So we face an apparent contradiction, which we will resolve in the present paper. The resolution turns out to be what one might characterize as a holomorphic anomaly in the

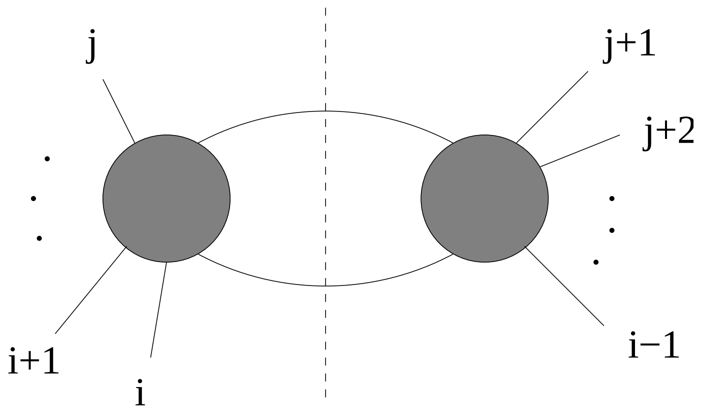

Figure 3: "Cut" diagram. Left and right tree-level MHV amplitudes are on-shell. Internal lines represent the legs coming from the "cut" propagators.

scattering amplitudes, or in the differential equations that one uses to analyze them. The "cut" diagram of figure 3 generates an amplitude that seems to be manifestly supported on a pair of lines, since the tree level MHV amplitude to the left or right of the cut is supported on a line. However, our criterion for asserting that an amplitude is supported on a pair of lines is that it should be annihilated by certain differential operators; when we act on the cut diagram with these operators, we get a non-zero result because of a certain holomorphic anomaly. The anomaly arises when one of the internal lines is collinear with an adjacent external line; this can occur for generic on-shell external momenta.

We think that the best summary of the facts is that the cut amplitude is indeed supported on a pair of lines, and the subtlety is in the use of the differential equations as a criterion for investigating this. The same should be true of the full amplitude, since the amplitudes are cut-constructible. The contributions of figure 1c found in [5] (in theories with varying amounts of supersymmetry) are thus no longer necessary, and the one-loop amplitudes, even with reduced supersymmetry, may potentially be generated by simple twistor-string theories.

In the next section, we explain the basic idea of the holomorphic anomaly and show that it produces a contribution to the cut with the structure of figure 1c.

2. The holomorphic anomaly

We will consider tree-level scattering of n gluons, labeled by $i = 1, \ldots, n$, with momenta $p_i^{a\dot{a}} = \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}$. We use spinor notation with conventions described in [1].

We focus on MHV amplitudes in which two gluons, say j and k, have negative helicity, while the others have positive helicity. We consider a subamplitude in which the gluons are in the cyclic order $123\ldots n$ and accordingly the group theory factor is $\mathrm{Tr}\ T_1 T_2 \ldots T_n$. This subamplitude (with the group theory factor and a momentum-conserving delta function $-ig^{n-2}\delta^4(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}})$ omitted) is

$$A(\lambda_i, \widetilde{\lambda}_i) = \langle \lambda_j, \lambda_k \rangle^4 \prod_{m=1}^{n} \frac{1}{\langle \lambda_m, \lambda_{m+1} \rangle}\,. \tag{2.1}$$

The fact that the amplitude is a function only of λ and not $\widetilde{\lambda}$ is described by saying that it is holomorphic. The terminology reflects the fact that for real momenta in Minkowski space, $\widetilde{\lambda}$ is plus or minus the complex conjugate of λ (the sign distinguishes initial and final particles).

The holomorphy was interpreted in [1] to mean that the n gluons in this scattering amplitude are supported on a line in twistor space. This collinearity was also expressed in terms of a differential equation. For this, we associate with a particle of momentum $p^{a\dot{a}} = \lambda^a\widetilde{\lambda}^{\dot{a}}$ its twistor coordinates $Z^I = (\lambda^a, \mu^{\dot{a}})$, where $\mu_{\dot{a}} = -i\partial/\partial\widetilde{\lambda}^{\dot{a}}$. The condition that particles i_1, i_2, and i_3 are collinear in twistor space is that $\epsilon_{IJKL}Z^I_{i_1}Z^J_{i_2}Z^K_{i_3} = 0$ for all L. For example, if we take $L = \dot{a}$, the condition is that $\langle\lambda_{i_1},\lambda_{i_2}\rangle\mu_{i_3}+\langle\lambda_{i_2},\lambda_{i_3}\rangle\mu_{i_1}+\langle\lambda_{i_3},\lambda_{i_1}\rangle\mu_{i_2} = 0$. Expressing μ in terms of $\partial/\partial\widetilde{\lambda}$, the differential operator that should annihilate a scattering amplitude that is supported on a line is

$$F_{i_1i_2i_3} = \langle\lambda_{i_1},\lambda_{i_2}\rangle\frac{\partial}{\partial\widetilde{\lambda}_{i_3}} + \langle\lambda_{i_2},\lambda_{i_3}\rangle\frac{\partial}{\partial\widetilde{\lambda}_{i_1}} + \langle\lambda_{i_3},\lambda_{i_1}\rangle\frac{\partial}{\partial\widetilde{\lambda}_{i_2}}. \tag{2.2}$$

The amplitude (2.1) is a function of the λ's only, so it appears to be manifestly annihilated by $F_{i_1i_2i_3}$, for all i_1, i_2, and i_3.

This is so for generic momenta, but there actually is a delta function contribution when two adjacent gluons (in the cyclically ordered chain $123\ldots n$) become collinear. To see this, we first rewrite $\widetilde{\lambda}$ as $\overline{\lambda}$, as is appropriate for real momenta.[2] The differential operator that should annihilate the amplitudes is then

$$F_{i_1i_2i_3} = \langle\lambda_{i_1},\lambda_{i_2}\rangle\frac{\partial}{\partial\overline{\lambda}_{i_3}} + \langle\lambda_{i_2},\lambda_{i_3}\rangle\frac{\partial}{\partial\overline{\lambda}_{i_1}} + \langle\lambda_{i_3},\lambda_{i_1}\rangle\frac{\partial}{\partial\overline{\lambda}_{i_2}}\,. \tag{2.3}$$

It seems that this operator does annihilate the MHV amplitude $A(\lambda_i,\overline{\lambda}_i)$, because this amplitude is holomorphic in the λ_i. But we must be careful; this function actually has poles, and these poles lead to delta functions when we act with the differential operators.

The basic fact that gives rise to the delta functions is that, for spinor variables λ and λ',

$$d\overline{\lambda}^{\dot{a}}\frac{\partial}{\partial\overline{\lambda}^{\dot{a}}}\frac{1}{\langle\lambda,\lambda'\rangle} = 2\pi\overline{\delta}(\langle\lambda,\lambda'\rangle)\,. \tag{2.4}$$

(Here, for a complex variable z, $\overline{\delta}(z) = -id\overline{z}\delta^2(z)$.) When this formula is inserted in the evaluation of $F_{i_1i_2i_3}A$, we get not zero, but rather delta function contributions when two adjacent gluons are collinear, that is, when they obey $\langle\lambda,\lambda'\rangle = 0$. This does not seem so important for tree diagrams, where for generic initial and final momenta, this collinearity does not arise. But the delta functions are significant when the tree-level MHV vertices are used to generate subamplitudes in a larger picture such as that of figure 3, where we must integrate over momenta of internal gluons, which may become collinear with one of the external gluons. (The integration region in computing the cut is compact, and there

[2] Even if one does not want the momenta to be real in Lorentz signature, to carry out the one-loop integrals, one needs an integration contour in which $\widetilde{\lambda}$ is a non-holomorphic function of λ; any such contour will lead to a similar result.

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are no singularities other than the collinear singularities we consider momentarily, so these singularities are the only source of an anomaly.)

In figure 3, for generic initial and final states, no two external momenta are collinear, and energy-momentum conservation does not permit the two internal lines — represented by "cut" propagators in the diagram — to be collinear. The important case, therefore, is that an external gluon is collinear with one of the internal gluons. Moreover, because of the form of the MHV tree amplitude A, a pole only arises if this collinearity involves an external and internal gluon that are adjacent in one of the vertices. In figure 3, we consider a configuration in which external particles $i, i+1, \ldots, j$ (for some i and j) couple to one MHV vertex and external particles $j+1, j+2, \ldots, i-1$ couple to the other vertex. Each vertex thus contains a chain of external gluons as well as the two internal gluons. A pole in the MHV amplitude associated with the vertex, leading to a delta function in the differential equations, arises if one of the gluons at the end of one of the chains (gluon i or j at one vertex, or gluon $j+1$ or $i-1$ at the other) is collinear with the adjacent internal gluon.

Whatever the external momenta may be, energy-momentum conservation allows any specified internal gluon to be collinear with any specified initial or final state gluon. This is obvious in the center of mass frame, where energy-momentum conservation fixes the energy of the internal gluons (each has half the total center of mass energy), while their directions of spatial motion are opposite but otherwise arbitrary. Hence, either internal gluon may propagate in any specified direction.

Notice, however, that once we require one internal gluon to be collinear with a given external gluon, the internal momenta in the loop are completely determined. So, for generic external momenta, we do not have the freedom to make *each* internal gluon collinear with one of the external gluons.

From this discussion, it follows that if we act on the cut amplitude B of figure 3 with $F_{i_1 i_2 i_3}$, where gluons i_1, i_2, and i_3 couple to the same MHV vertex, we will get zero if none of the particles i_1, i_2, or i_3 are at the end of one of the chains. If one of i_1, i_2, or i_3 is at the end of a chain, the action of F will give a delta function, which then, upon integrating over the internal momenta, will give a nonzero result. So for these choices of the gluons, $F_{i_1 i_2 i_3}$ does not annihilate the amplitude.

Since $F_{i_1 i_2 i_3} B = 0$, where i_1, i_2, and i_3 are disjoint from the ends of the chains, it follows that all of the gluons at a vertex that are not at an end of their respective chain are supported on a line. Thus, as there are two vertices, all gluons except the endpoint gluons $i, j, j+1$, and $i-1$ are supported on a union of two lines. Moreover, all but one of the endpoint gluons are supported on the two lines. To prove this, we simply write the integral over internal momenta in the cut amplitude as a sum of integrals over subregions each of which only contains one pole. This is possible for a reason observed earlier: for generic external momenta, energy-momentum conservation does not allow two distinct endpoint gluons, such as particles i and j, to each be collinear with an internal gluon. Containing only one pole, each subregion contributes to the amplitude a term in which at most one gluon is not contained in the pair of lines. So the full amplitude is a sum of terms in each of which, from the standpoint of the differential equations, all gluons but one are contained in the union of two lines.

We can express this reasoning in terms of differential equations by observing that the contribution to the cut from the diagram of figure 3 is annihilated by suitable products of F's, for example

$$F_{ii_1i_2}F_{jj_1j_2}B = 0\,, \tag{2.5}$$

where i, i_1, i_2 and likewise j, j_1, j_2 each couple to the same vertex. This holds, again, because anomalies involving the different F's come from disjoint regions of the integration over internal momenta. Eq. (2.5) is interpreted to mean that B is a sum of terms annihilated by one F or the other, that is, a sum of terms in which either particle i or particle j is contained in the union of the two lines.

The holomorphic anomaly thus leads to precisely the situation that actually was found in [5]: if one uses the differential equations as a criterion for collinearity, then $n-1$ of the n gluons are contained in two chains of consecutive gluons, with each chain being supported on a line in twistor space. The remaining gluon is generically not contained in this union of lines. We think, however, that it is most natural to describe the cut amplitude of figure 3 as being supported on a pair of lines, because in twistor-string theories we expect this kind of amplitude to be generated by contributions that intuitively are supported on a pair of lines. We prefer to interpret the phenomenon we have found as a subtlety in the use of the differential equations as a criterion for collinearity.

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References

[1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, hep-th/0312171.

[2] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, *J. High Energy Phys.* **04** (2004) 012 [hep-th/0402016];
R. Roiban and A. Volovich, *All conjugate-maximal-helicity-violating amplitudes from topological open string theory in twistor space*, hep-th/0402121;
R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, *Phys. Rev.* **D 70** (2004) 026009 [hep-th/0403190].

[3] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *J. High Energy Phys.* **09** (2004) 006 [hep-th/0403047].

[4] S. Gukov, L. Motl and A. Neitzke, *Equivalence of twistor prescriptions for super Yang-Mills*, hep-th/0404085.

[5] F. Cachazo, P. Svrcek and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, *J. High Energy Phys.* **10** (2004) 074 [hep-th/0406177].

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[6] A. Brandhuber, B. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from mhv vertices*, hep-th/0407214.

[7] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop N point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].